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Bohr–Sommerfeld star products

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Abstract

We relate the Bohr–Sommerfeld conditions to formal deformation quantization of symplectic manifolds by classifying star products adapted to some Lagrangian submanifold L, i.e. products preserving the classical vanishing ideal $\mathcal{I}_L$ of L up to $\mathcal{I}_L$-preserving equivalences.

Keywords: Bohr–Sommerfeld conditions; Deformation quantization; Adapted star product; Star representation; Lagrangian submanifold; Integrable system; Maslov index

1. Introduction and motivation

1.1. Reminder on Bohr–Sommerfeld conditions

Let $L \overset{i_L}{\hookrightarrow} T^*Q$ be a Lagrangian submanifold of some cotangent bundle $T^*Q \overset{\pi}{\to} Q$ with respect to the standard symplectic structure $\omega := -d\theta$ given by the canonical form $\theta := T^*\pi$, and μ its Maslov class. Then L has to satisfy the prequantization condition

$$\frac{1}{2\pi\lambda} i_L^*\theta - \frac{\pi}{2}\mu \in H^1_{dR}(L, \mathbb{Z}). \tag{1}$$

in order to be the microsupport of some λ-oscillatory distribution on Q (see Appendix for references), where $H_{dR}(., \mathbb{Z})$ denotes the integral de Rham classes.

In particular, if $L_E := H^{-1}(E)$ is a Liouville torus of some semiclassical integrable system with classical moment map $H : T^*Q \to \mathbb{R}^n$ and vanishing subprincipal form[1] κ, then the condition (1) coincides up to higher orders $O(\lambda^1)$ with the Bohr–Sommerfeld conditions

$$\frac{1}{2\pi\lambda} i_L^*\theta - \frac{\pi}{2}\mu + \kappa + O(\lambda) \in H^1_{dR}(L, \mathbb{Z}) \tag{2}$$

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[1] Let s_i be the second-order part of the Weyl symbols of the $\hat{H}_i$, then by definition $\kappa.\mathcal{X}H_i = -s_i$.

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for the existence of a joint asymptotic eigenvector of the system.

Recall here that a semiclassical integrable system is a maximal set of commuting λ-pseudodifferential operators $\hat{H}_1, \ldots, \hat{H}_n$ whose principal symbols $H = (H_1, \ldots, H_n)$ are independent almost everywhere. Then the compact fibers L_E of H restricted to regular values B are Lagrangian submanifolds with a transitive locally free $\mathbb{R}^n$ action $\phi : \mathbb{R}^n \ni t \mapsto \exp \mathcal{X} \langle t, H \rangle$ generated by the Hamiltonian vector fields $\mathcal{X} H_i := \omega^{-1} dH_i$, hence they are tori $L_E \cong \mathbb{R}^n / \ker \phi|_{L_E} \cong \mathbb{R}^n / \mathbb{Z}^n$. This action is linearized in action-angle coordinates; similarly, the semiclassical system is microlocally unitary equivalent to the linearized system, such that the microlocal solutions (oscillatory constants) form a local system whose triviality is the Bohr–Sommerfeld condition (2), cf. [28].

1.2. Problems with interpretations in deformation quantization

The Bohr–Sommerfeld condition (2) makes precise the original notion of quantization as ad hoc discretization of classical spectra. However, it is undefined in formal deformation quantization, a notion isolating the transition from commutative to noncommutative algebras underlying any quantization concept (see Appendix, (18) for references). Here the deformation parameter λ is formal, one thus has consider convergent deformations (of a subalgebra) over some base including $[0, \hbar]$ to recover the meaning of (1). However, we can try to extract the prequantization class in (1) from deformation quantization as formal class. This has been done in [25] by formalizing the symbol calculus of oscillatory distributions. Here, we proceed differently:

1.3. Main results and outline

Motivated by oscillatory symbol calculus as well, we first consider $\star$-representations on line bundles over $L \subset X$. Then we look for star products inducing a canonical representation on L, namely, we establish a bijection between the classes of deformation quantizations of X preserving the classical vanishing ideal $\mathcal{I}_L \subset C^\infty(X)[[\lambda]]$ of L up to $\mathcal{I}_L$-preserving equivalences and those of formal deformations of the symplectic form viewed as relative class. This is done in Section 3 by parametrizing adapted Fedosov star products. Then in Section 4 we consider the induced intertwiners on the quotients as formal analogues of L-deformations in order to explain the coincidence of (1) with adapted classes in the lowest order. Finally, we sketch relations to the Maslov index and the symbol calculus of oscillatory distributions. The Appendix provides the motivating background and sets up some (standard) notations.

2. Representations on line bundles

Let $\star$ be a star product on a symplectic manifold X and E some vector bundle over a Lagrangian submanifold $L \subset X$. A $\star$ representation on E means a $\star$-module structure on $\Gamma(E)[[\lambda]]$ such that $\star$ acts by $\mathbb{C}[[\lambda]]$-linear differential operators.

Finally, define a deformed flat line bundle on L by the requirement that its sheaf of local sections is locally isomorphic to the constant sheaf $\exp(\mathbb{C}[[\lambda]])_L$ of the units $\exp(\mathbb{C}[[\lambda]]) \subset \mathbb{C}[[\lambda]]$. The isomorphism classes of such bundles are then given by $\check{H}^1(X, \exp(\mathbb{C}[[\lambda]])_L)$ like in the undeformed case of flat line bundles. This group now acts naturally on $\star$ representations on E thanks to their $\mathbb{C}[[\lambda]]$-linearity. The action turns out to be free and transitive:

Lemma 1. *$\star$ is representable on some complex line bundle E over a Lagrangian submanifold L if and only if the image $c_1^{\mathbb{R}}(E)$ of its Chern class $c_1(E)$ under $^{\mathbb{R}} : H^2(L, \mathbb{Z}) \to H^2_{dR}(L)$ coincides with the restriction of the equivalence class $[\star]$ of $\star$:*

$$c_1^{\mathbb{R}}(E) = i_L^*[\star]. \tag{3}$$

The space $\mathcal{M}_E$ of isomorphism classes of $\star$ representations on E identifies with the group of deformed flat line bundles $\check{H}^1(L, \exp(\mathbb{C}[[\lambda]])_L)$ via its natural action on $\mathcal{M}_E$.

Proof. This is a direct consequence of Bordemann's classification: Consider the restriction of $\star$ to some tubular neighborhood of L which we may identify with a neighborhood W of the zero section in $(T^*L, -d\theta)$ by some Weinstein isomorphism (cf. [7, Th. 4.19]). Then by [5, Th. 3.3] the product $\star|W$ is equivalent to a standard ordered product $\star_B$, $[B] = i_L^*[\star]$, whose representation $\bullet$ on some complex line bundle E on L must locally on some contractible set U_i look like

$$(\pi^*\psi \bullet \phi) = \psi\phi, \qquad (\theta.\hat{X}) \bullet \phi|U_i = -2\lambda(X - \frac{A_i.X}{2\lambda})\phi|U_i \tag{4}$$

for any $\phi \in C^\infty(L)$ and any vector field $X \in \Gamma(TL)$ with canonical lift $\hat{X} \in \Gamma(T_0TL)$. Here the A_i are determined by $dA_i = B|U_i$ up to some coboundary dS_i, which determines $\bullet|U_i$ up to some local intertwiner (gauge equivalence) $\phi|U_i \mapsto e^{S_i}\phi|U_i$. Thus (3) must hold, which determines the representation up to isomorphism classes of formal flat connections

$$\lambda H^1_{dR}(L)/2\pi i H^1_{dR}(L;\mathbb{Z}) + \lambda^2 H^1_{dR}(L)[\![\lambda]\!]$$

on the chosen torsion bundle in $\ker^{\mathbb{R}}$. □

Remark 1. Consider two line bundles E, E' over L. If E is a $\star$ representation and $c_1(E') - c_1(E) \in \operatorname{im}(i_L^* : H^2(X,\mathbb{Z}) \to H^2(L,\mathbb{Z}))$ then one may obtain a $\star'$ representation on E' via Rieffel induction:

Namely, it was shown in [8] that $\operatorname{Pic}(X) \ltimes \operatorname{Aut}(X,\omega)$ acts transitively on the Morita equivalent equivalence classes of $\star$ such that $[(\mathcal{L},\phi)\cdot\star] = [\star] + c_1^{\mathbb{R}}(\mathcal{L})$. Here the symplectomorphism $\phi \in \operatorname{Aut}(X,\omega)$ acts by pull back while the action of the line bundle $\mathcal{L}$ on X arises by deforming its transition 1-cocycles to $\star$-cocycles defining an equivalence $(\star',\star)$-bimodule $\boldsymbol{\mathcal{L}}$, cf. [8, sec. 4.2]. Hence $\boldsymbol{\mathcal{L}} \otimes_\star E$ indeed defines a $\star'$ representation on $i_L^*\mathcal{L} \otimes E$.

3. Adapted star algebras

Recall that a star product $\star$ on X is called adapted to some Lagrangian submanifold $L \subset X$ if the classical vanishing ideal $\mathcal{I}_L := \{f \in \mathcal{A} |\ f|L = 0\}$ of L in $\mathcal{A} := C^\infty(X)$ remains a $\star$-left ideal $\boldsymbol{\mathcal{I}}_L = \mathcal{I}_L[\![\lambda]\!]$ and thus induces a $\star$ representation $\mathcal{A}[\![\lambda]\!]/\boldsymbol{\mathcal{I}}_L$ on L. Such products are formal series of one chains of the subcomplex $K_{\mathcal{I}_L} := \left\{C \in \mathcal{C}^\bullet(\mathcal{A};\mathcal{A}) |\ C(\mathcal{A}^{\otimes\bullet} \otimes \mathcal{I}_L) \subset \mathcal{I}_L\right\}$ of the differential Hochschild complex[2] of $\mathcal{A}$ fitting into an exact sequence

$$K_{\mathcal{I}_L} \hookrightarrow \mathcal{C} \twoheadrightarrow \mathcal{C}(\mathcal{A};\mathcal{C}^1(\mathcal{I}_L;\mathcal{A}/\mathcal{I}_L))[-1]$$

by [3, Prop. 2.2]. Its corresponding long exact cohomology sequence decouples into short exact sequences isomorphic to the one defining relative de Rham forms

$$\Omega(X,L) \hookrightarrow \Omega(X) \overset{i_L^*}{\twoheadrightarrow} \Omega(L). \tag{5}$$

This was shown locally in [3, Th. 2.4] for the ω-corresponding multivector fields via Koszul resolutions, from which the global case can be deduced by the degeneration of the local-to-global spectral sequence at $E_2^{pq} = H^p(\Omega^q(X,L)_X) = \Omega^q(X,L)\delta_{p0}$. However, we will not use this "adapted HKR theorem", although together with (21) it implies directly the following Lemma: Denote by δ the connecting homomorphism of the long exact sequence

$$\cdots H^\bullet_{dR}(X) \to H^\bullet_{dR}(L) \overset{\delta}{\to} H^{\bullet+1}_{dR}(X,L) \to H^{\bullet+1}_{dR}(X)\cdots \tag{6}$$

associated to the relative de Rham sequence (5). Then we have:

Lemma 2. *Let S be an equivalence between two star products $\star,\star'$ on X both adapted to L. If $\delta H^1_{dR}(L) = \{0\} \subset H^2_{dR}(X,L)$, then S is adapted, i.e. preserves $\boldsymbol{\mathcal{I}}_L$ and hence provides an equivalence[3] of represented algebras $(\star,\star/\boldsymbol{\mathcal{I}}_L) \sim (\star',\star'/\boldsymbol{\mathcal{I}}_L)$.*

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[2] Recall that the differential Hochschild complex $\mathcal{C}^\bullet(\mathcal{A},M)$ of $\mathcal{A}$ with values in some $\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}$-representation M consists of M-valued differential $\mathbb{C}$-multilinear operators $\mathcal{C}^{k-1}(\mathcal{A},M) := \operatorname{Hom}^{\mathrm{diff}}_{\mathbb{C}}(\mathcal{A}^{\otimes k},M)$ with coboundary $bC(f_0,\ldots,f_n) = f_0C(f_1,\ldots,f_n) + \sum_i(-1)^{i+1}C(f_0,\ldots,f_if_{i+1},\ldots f_n) + (-1)^{n-1}C(f_0,\ldots f_{n-1})f_n$.

[3] Following [5, Prop 3.4] we call two represented algebras $(\mathcal{A},\mathcal{H}), (\mathcal{A}',\mathcal{H}')$ equivalent if there is an isomorphism $S : \mathcal{A} \to \mathcal{A}'$ and an "intertwiner" $T : \mathcal{H} \overset{\sim}{\to} \mathcal{H}'$ such that $T(f\cdot\psi) = S(f)\cdot T\psi$.

Proof. Suppose that $\star, \star'$ are already identical up to $O(\lambda^k)$ thanks to an equivalence adapted up to $O(\lambda^k)$. Then by (21) and (22) one has Lichnerowicz's equation

$$\star' - \star = bT + d\alpha(\mathcal{X}., \mathcal{X}.) \bmod O(\lambda^{k+1}) \tag{7}$$

for some $\alpha \in \Omega^1(X)$, where b is the Hochschild coboundary of the undeformed product '$\cdot$'. As decomposition into symmetric and antisymmetric parts, both summands have to be adapted by induction hypothesis, so $di_L^*\alpha = 0$. Now by the assumption there exists some relative primitive of $d\alpha$ we may suppose to be α itself.

Then the equivalence $S_\alpha := 1 + \lambda^{k-1}\alpha\mathcal{X}$ is adapted and turns the difference into a coboundary

$$S_\alpha(\star') - \star = \lambda^k b(T - [T, \alpha\mathcal{X}]) =: \lambda^k bT' \bmod O(\lambda^{k+1})$$

which is adapted if and only if T' is, since $bT'(\mathcal{I}_L, \mathcal{I}_L) \subset T'(\mathcal{I}_L^2) + \mathcal{I}_L$. Thus $S' := (1 + \lambda^k T') \circ S_\alpha$ is an adapted equivalence modulo $O(\lambda^{k+1})$. $\square$

3.1. Reminder on Fedosov's construction

There is a construction of natural[4] Weyl type star products due to Fedosov [13] which has a natural interpretation in formal geometry context as observed by [24]:

Let $\widehat{\mathbb{R}^{2n}} := (\mathbb{C}[[\xi^1, \ldots, \xi^{2n}; \lambda]], *_W)$ denote the formal Weyl algebra

$$f *_W g := \mu_0 \left(\exp\left(\frac{\lambda}{2i} \omega^{ij} \partial_i \otimes \partial_j \right) f \otimes g \right), \tag{8}$$

where μ_0 denotes the standard multiplication. $*_W$ is $\mathbb{Z}$-graded by $\deg \lambda = 2$, $\deg \xi = 1$ and invariant under the linear symplectic group $Sp(n, \mathbb{C}) \cong \operatorname{ad} \widehat{\mathbb{R}^{2n}}_2$ generated by quadratic forms in $\widehat{\mathbb{R}^{2n}}_2$, where as usual the subscript denotes the degree.

One now considers infinitesimal patching of local algebras on X, i.e. the bundle P of isomorphisms $jet_x(X)[[\lambda]] \xrightarrow{\sim} \widehat{\mathbb{R}^{2n}}$, $x \in X$, deforming the real symplectic frame bundle $Sp(X) = \bigcup_x (T_x X, \omega_x) \xrightarrow{\sim} (\mathbb{R}^{2n}, \omega_0)$. Then the natural isomorphism $\theta : T_x P \xrightarrow{\sim} \mathfrak{g} := \left\{ \frac{i}{\lambda} \operatorname{ad} f \middle| \, f \in \widehat{\mathbb{R}^{2n}}, \Im(f) = 0 \bmod O(\lambda) \right\}$ is a flat $\mathfrak{g}$-valued connection, i.e. a Sp-equivariant 1-form such that its composition with the $\mathfrak{sp}$-action is the inclusion $\mathfrak{sp} \to \mathfrak{g}$ and the curvature $d\theta + \frac{1}{2}[\theta, \theta]$ vanishes. Now any reduction of the structure group given by a section r of $P \to Sp(X)$ induces a flat connection $\nabla_F = d + \frac{i}{\lambda} \operatorname{ad} r^*\theta$ on the associated Fedosov bundle $\mathcal{W} := Sp(X) \times_{Sp(n)} \widehat{\mathbb{R}^{2n}}$. By [24], its constant sections $\ker \nabla_F$ are isomorphic to a star product algebra on X via r whose characteristic class is represented by the pullback by r of the curvature of the lift of θ to a connection with values in the central extension $\widehat{\mathbb{R}^{2n}}$ of $\mathfrak{g}$.

More explicitly, any Fedosov connection ∇_F has to start with an equivariant degree -1 square zero differential fixed as

$$-\delta := -\frac{i}{\lambda} \operatorname{ad} r^*\theta_1 := -\frac{\partial}{\partial \xi^i} \otimes dx^i.$$

Further the λ-independent degree zero part $\frac{\partial}{\partial x^l} + \frac{i}{\lambda} \operatorname{ad} \Gamma^l_{jk} \xi^j \xi^k$ represents a torsion free symplectic connection ∇. Thus the simplest Fedosov connection is of the form $\nabla_F = -\delta + \nabla + \frac{i}{\lambda} \operatorname{ad}_{*_W} \gamma$ for some $\gamma \in \Omega^1(X; \mathcal{W}_{\geq 3})$.

However, to get natural products of different order type, one needs equivalent fiberwise degree 0 products $* = \mu_0 \exp(\frac{\lambda}{2i} \mu_{ij} \frac{\partial}{\partial \xi_i} \otimes \frac{\partial}{\partial \xi_j}) = e^{\lambda S}(*_W)$, where S is a fiberwise degree -2 differential operator with fiberwise constant coefficients, which in turn requires Fedosov derivations of the generalized form (cf. [23])

$$\nabla_F = -\delta + D + \frac{i}{\lambda} \operatorname{ad} \gamma, \tag{9}$$

[4] Following [14], we call a star product $\star$ natural if $\star_k$ is of differentiation order $\leq k$ in each argument.

where D is a degree 0 $*$-derivation with $D|1 \otimes \Omega(X) = d$, $D^2 = -\frac{i}{\lambda}\,\mathrm{ad}\,R$ and $[D, \delta] = \frac{i}{\lambda}\,\mathrm{ad}\,T$ for totally covariant curvature and torsion tensors R and T. Namely, we may and will take the torsion free derivation

$$D = \nabla + \frac{i\lambda}{2}[\nabla, S]. \tag{10}$$

Note that at this point it might become more natural to work with the deformation of the symmetric algebra ST^*X induced by $(\mathcal{W}, *)$ and the canonical isomorphism $\mathcal{W}/\lambda\mathcal{W} \cong ST^*X$, as $(\mathcal{W}, *)$ is no longer given as associated bundle.

Now, for any formal closed two form $\Omega \in Z^2_{dR}(X)[[\lambda]]$ and any $s \in \mathcal{W}_{\geq 4}$ with trivial central part $\sigma(s) = 0$ the equations

$$\delta\gamma = D\gamma - \frac{1}{\lambda}\gamma * \gamma + R + \Omega, \quad \delta^{-1}\gamma = s \tag{11}$$

determine γ such that ∇_F will be a Fedosov connection whose constant sections $(\ker \nabla_F, *)$ are naturally isomorphic to a natural star product $\star_F$ of class $[\star_F] = [\lambda^{-1}\omega + \Omega]$. More precisely, ∇_F extends to an acyclic superderivation on $\Omega(X, \mathcal{W})$ with contracting homotopy

$$\nabla_F^{-1}\alpha := -\delta^{-1}\frac{1}{1 - [\delta^{-1}, D + \frac{i}{\lambda}\,\mathrm{ad}(\gamma)]}$$

(in the sense of the geometric series), where δ^{-1} is defined as homotopy on the center $(\delta^{-1}\delta + \delta\delta^{-1})\alpha = \alpha - \sigma(\alpha)$. Then the isomorphism $\star_F \cong \ker \nabla_F \cap \mathcal{W}$ is given by the restriction of σ with inverse $\tau(f) := f - \nabla_F^{-1} df$.

Using [23, 1.3.25,27], one checks that the parametrization by $\nabla, \Omega, *, s$ is redundancy free for fixed $*$. Moreover, it is conjecture that any natural star product arises as generalized Fedosov star product.

Proposition 1. *A generalized Fedosov star product $\star_F$ on T^*L given by (9)–(11) is adapted to L if and only if its construction data $\nabla, \Omega, *, s$ are adapted, i.e:*

i. *∇ restricts to a connection on L. By the absence of torsion this is equivalent to L being totally geodesic.*
ii. *L is Lagrangian for the "deformed symplectic structure" $\omega + \Omega$*
iii. *$s \in I_{TL}$, where $I_{TL} = (TL^0)[[\lambda]]$ is the ($\mathbb{C}[[\lambda]]$-extended) fiberwise vanishing ideal of TL generated by the annulator $TL^0 \subset ST^*X$ of TL.*
iv. *The fiberwise product $*$ is adapted to I_{TL}, i.e., in symplectic fiber coordinates q^i, p_j over L such that $I_{TL} = (p_1, \ldots, p_n)$ we have $S = \frac{\lambda}{2i}\frac{\partial}{\partial p_j}\frac{\partial}{\partial q^j}$ and $* = \mu_0 \exp(\frac{\lambda}{2i}\frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q^j})$.*

Proof. Let $I_F^{s,p} \subset \Omega^p(X, \mathcal{W}_s)$ be the subspace of adapted forms whose restriction to $\wedge^p TL$ has per definition values in I_{TL}. Now, if all construction data are adapted, then $\gamma \in I_F$, since the same holds for all summands in (11). Here the only nonobvious term is $[\nabla, S]$, where the claim holds if ∇ is the homogeneous adapted connection ∇^0 used in [4], then it follows in general by $\nabla - \nabla^0 \in \frac{i}{\lambda}\,\mathrm{ad}\, I_F^{2,1}$. Thus ∇_F preserves I_F such that $\tau\mathcal{I}_L = \ker \nabla_F \cap I_{TL}$, hence $\star_F$ is adapted. – Vice versa:

i. Let σ' denote the projection $\Omega(X, \mathcal{W}) \to \Omega(X, \mathbb{C})[[\lambda]]$. We have $\sigma'\delta\tau f = \delta\delta^{-1}df = df$. Now $df|TL = 0$ for any $f \in \mathcal{I}_L$, so adaptivity implies

$$\sigma'\delta(X * \tau I)|TL = 0 \quad \forall X \in \ker \nabla_F, I \in \mathcal{I}_L \tag{12}$$

with $\tau f = f + Df + D^2 f$ modulo deg ≥ 3, where $D = [\delta^{-1}, \nabla]$ here and in the following denotes the induced symmetric covariant connection on $\mathcal{W}$. Thus (12) implies

$$0 = (X *_1 \delta D^2 I)|TL \tag{13}$$

for some X of total degree 1. By iv $X*$ only differentiates along L, thus for any vector fields X, Y tangential to L one has $0 = D^2 I(X, Y)|L = dI(\nabla_X Y + \nabla_Y X)|L = dI(2\nabla_X Y)|L$ by the absence of torsion and $dI.[X, Y]|L = 0$, thus $\nabla_X Y$ must be tangential to L.

iv. Since $\tau^k = D^k$ modulo lower-order differentiation along Z, we have

$$f(\star_F)_k I = \mu_{IJ}\partial^I f \partial^J I \text{ modulo lower-order differentiation} \tag{14}$$

for some nondegenerate tensor $\mu \in \Gamma(\mathcal{W}_k \otimes \mathcal{W}_k)$. In particular, adaptivity and (20) imply $\mu_{ij} = \frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q^j}$ for $k = 1$.
ii and iii Allow $s \in \mathcal{W}_{\geq 3}$ to make Ω locally redundant as in [23, 1.3.27], and denote by $\tau(s)$ the dependence of τ on s. Then by Lemma 2 and adaptivity of $\star_F$ for adapted construction data including $s = 0$ we must have $\sigma(\tau(s)^{2k} - \tau(0)^{2k})\mathcal{I}_L \subset \mathcal{I}_L$. Now s^k first occurs deg-inductively in τ^{2k-2} as $[\delta s^k f, \tau^{k-1} I]_{*_{k-1}}$, hence $\tau^{k-1} I *_{k-1} \delta s^k f | TL = 0$ for all $I \in \mathcal{I}_L$, thus s is adapted by standard order of $*$. □

Remark 2. Note that Ω_k enters explicitly as $\star_{\nabla, \Omega+\lambda^k \Omega_k, \circ, s} = \star_{\nabla, \Omega, \circ, s} + \lambda^{k+1} \Omega_k(\mathcal{X}., \mathcal{X}.) \bmod O(\lambda^{k+2})$.

Now identify some zero section environment of T^*L with some neighborhood of $L \subset X$ via some Weinstein isomorphism. Then note that there are no obstructions for the extension of the data to all of X: For the fiberwise data this is clear from partition of unity, for the connection this follows from the description of the space of symplectic connections as sections of the fiber bundle $J^1 Sp(X)/Sp$ having contractible fibers $F := J^1_0(\mathbb{R}^{2n}, Sp)_1 = \mathbb{R}^{2n} \times \mathfrak{sp}$ (see [18]), hence the obstruction classes $H^i(X, L; \pi_{i-1}(F))$ vanish. (Note that this argument ignores torsion, which is possible due to [23, prop 1.3.31].)

Now by (3) the characteristic class $[\star]$ of an adapted star product must have a relative representative $\eta \in Z^2_{dR}(X, L)((\lambda))$. For any relative cocycle η the above construction yields an adapted Fedosov star product $\star_F$ with $\eta = \lambda^{-1}\omega + \Omega$, and two adapted Fedosov products differing only in their 2-forms by $\lambda^k d\alpha \bmod O(\lambda^k)$ are equivalent through $S_\alpha := 1 + \lambda^{k-1}\alpha.\mathcal{X} \bmod O(\lambda^k)$ by Remark 2 and (7), which covers the cohomology in (7). Since S_α is adapted if and only if $i_L^*\alpha = 0$, i.e. $[d\alpha] = 0 \in H^2_{dR}(X, L)$, we obtain:

Theorem 1. *The adapted equivalence classes $[.]_L$ of L-adapted deformation quantizations are in bijection to relative formal $\frac{\omega}{\lambda}$-deformations*

$$\lambda^{-1}[\omega] + H^2_{dR}(X, L)[[\lambda]]$$

where the image and kernel of $H^2_{dR}(X, L)[[\lambda]]$ in the long exact sequence (6) correspond to the absolute class and the adapted classes therein, respectively. □

Remarks. 3. In the preprint [3] the generalized problem of deforming the adapted HKR quasiisomorphism to an adapted G_∞ resp. L_∞ morphism has been attacked locally, see as well [10].
4. As shown by Gromov [22, Th. 7.34], for an open manifold any de Rham cohomology class is representable by a symplectic form unique up to isotopy. This of course does not hold in the relative case, for instance the trivial class in $H^2_{dR}(\mathbb{R}^2, S^1)$ has no symplectic representative ω by $\int_{\mathrm{int} S^1} \omega \neq 0$. In fact, the same holds for any compact Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ by the above isotopy theorem and Gromov's nonexistence theorem of exact $L \subset \mathbb{R}^{2n}$ [22, Th. 13.5].
5. By the same method, we can construct star products adapted to transversal intersections of Lagrangian submanifolds or all the fibers of a regular Lagrangian fibration.

4. Relation to Bohr–Sommerfeld conditions

We now want to "deduce" the prequantum Bohr–Sommerfeld conditions from the picture

$$\begin{array}{ccccc} \dfrac{\frac{1}{\lambda} i_L^*\theta + H^1_{dR}(L)[[\lambda]]}{\mathrm{im} \frac{1}{\lambda} H^1_{dR}(X)[[\lambda]]} & \xhookrightarrow{\delta} & \frac{\omega}{\lambda} + H^2_{dR}(X, L)[[\lambda]] & \longrightarrow & \frac{\omega}{\lambda} + H^2_{dR}(X)[[\lambda]] \\ \updownarrow & & \updownarrow & & \updownarrow \\ \text{adapted classes inside an absolute class} & & \text{adapted classes} & & \text{absolute classes} \end{array} \tag{15}$$

following from Theorem 1.

Lemma 3. *Consider the group of equivalences $S_\alpha = e^{\lambda \alpha.\mathcal{X}}$ between adapted star products modulo adapted equivalences, which is homomorphically parametrized by $i_L^*[\alpha] \in H^1_{dR}(L)[[\lambda]]$. Then the action of S_α on equivalence classes of pairs* {(adapted star product, canonical representation)} *induces the action of the flat deformed line bundle E_α with holonomy $\pi_1(L) \ni \gamma \mapsto e^{i\lambda \int_\gamma \alpha}$ on representation classes $\mathcal{M}_0$ (Lemma 1).*

Indeed, in a Weinstein model T^*L near L with Darboux coordinates I, φ we can assume that $\alpha = \alpha_i d\varphi_i$ near L for $\alpha_i \in \mathbb{C}[[\lambda]]$, then S_α acts on contractible patches like the inner automorphisms $S_\alpha = e^{\lambda\alpha_i\{\varphi_i,\cdot\}} = e^{\lambda\alpha_i \operatorname{ad}_\star(\varphi_i)}$ on the standard ordered product $\star$ of T^*L. As inner automorphisms induce self-intertwiners of $\star/\mathcal{I}_L$, globally we get the desired action of E_α on $\mathcal{M}_0$. $\square$

Hence in the case of convergence, *integral* adapted equivalences $S_\alpha, \lambda i_L^*\alpha \in H^1_{dR}(L, \mathbb{Z})$ should provide intertwiners between formally in-equivalent representations which correspond to relative quantization conditions as follows:

4.1. Relative conditions

Restrict H to a regular torus fibration $H_{\text{reg}} = H|X_{\text{reg}}$ over B near L. Then there is a natural identification $\mathbf{R}^1 H_{\text{reg}}\mathbb{R} = T_B$ obtained as dual of the action differential $\gamma_i \mapsto dI_i := d\int_{\gamma_i}\theta$.

In particular, we can *canonically* identify small $\alpha_i[d\varphi_i] \in H^1_{dR}(L)$ with the translated torus $L_\alpha := H^{-1}_{\text{reg}}(\alpha_1, \ldots, \alpha_n)$ (in action coordinates with origin $0 = H(L)$) which equals the image $\operatorname{im}\alpha_i d\varphi \subset T^*L$ in the Weinstein model near L determined by the action-angle coordinates.

Now S_α maps the vanishing ideal $\mathcal{I}_L$ of L to that of the translated Lagrangian torus L_α: In fact, infinitesimally, this map corresponds to the isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{I}_L/I_L^2, \mathcal{A}/\mathcal{I}_L) \cong \Gamma(T_LX/TL) \cong \Omega^1(L) : \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_0 S_\alpha = \alpha.\mathcal{X} \mapsto [\omega^{-1}\alpha|L] \mapsto i_L^*\alpha$$

where $\mathcal{A} := C^\infty(X), \mathcal{I}_L := \ker i_L^* \subset \mathcal{A}$ are the classical $\mathcal{A}$-modules.

Hence for integral $i_L^*\lambda\alpha$, S_α intertwines the canonical $\star$ representations (D-modules) $\star/\mathcal{I}_L, \star/\mathcal{I}_{L_\alpha}$ on L and L_α. But such intertwiners correspond to joint $(\alpha_1, \ldots, \alpha_n)$-eigenspaces of H_{reg} by the standard D-module identity (cf. [17, Ch. 0])

$$\ker\left((H_{\text{reg},i} - \alpha_i) : \frac{\mathcal{A}}{\mathcal{I}_L} \to \left(\frac{\mathcal{A}}{\mathcal{I}_L}\right)^n\right) = \hom_D\left(\frac{D}{(H_{\text{reg},i} - \alpha_i)}, \frac{\mathcal{A}}{\mathcal{I}_L}\right).$$

Remark 6. Note that our reference Lagrangian $L \subset X_{\text{reg}}$ itself is always quantizable and in order to speak of nontrivial relative classes on X_{reg}, one has to identify the class $\alpha_i[d\varphi_i] \in H^1_{dR}(L)$ with $\alpha_i[d\varphi_i] \oplus 0 \in H^1_{dR}(-L \cup L_\alpha)$ giving a meaningful class in $H^2_{dR}(X_{\text{reg}}, -L \cup L_\alpha)$. Then by (1) and (2) the relative integrality conditions are indeed related to the the joint asymptotic spectrum of H (the Bohr–Sommerfeld conditions) by an embedding $H(X_{\text{reg}}) \to \mathbb{R}^n$ which is integral affine in leading order.

4.2. Bohr–Sommerfeld conditions

Similar to the embedding of relative spectra in Remark 6, "combining" the relative conditions with the formal picture (15) through Lemma 3 now leads to the following

Suggestion 1. *Let $\star$ be a deformation quantization of a symplectic manifold (X, ω) adapted to a Lagrangian submanifold L. Then the formal analogue of (the δ-image of) the prequantum Bohr-Sommerfeld class (1) is given by $[\star]_L$.*

The point of our approach to prequantum Bohr–Sommerfeld classes is that it reproduces the leading order $\frac{1}{\lambda}i_L^*\theta$ of (1) and (2) already on the formal algebraic level without involving WKB type methods. This gives another explanation of the independence of the leading-order conditions on the quantization itself. Moreover, it generalizes them to arbitrary symplectic manifolds where the integrality condition on relative classes implies one on absolute classes as required by geometric quantization.

On the other hand, the Maslov class is not visible in this approach. Let us note though that it can be easily extracted from adapted Fedosov star products as follows:

4.3. Maslov index

For the Maslov class μ to be defined consider an additional Lagrangian fibration π of X around L (the example in mind is of course the vertical fibration in the case of $X = T^*Q$). Let $\nabla(\star_F)$, $\nabla(\star'_F)$ be the symplectic connections inside the Fedosov connections of two Fedosov star products $\star_F, \star'_F$ adapted to L and the fibers of π respectively. On $T_L X$, we may further assume these connections to be unitary with respect to some compatible almost $\mathbb{C}$-structure on X, such that the difference $\nabla(\star_F) - \nabla(\star'_F)$ is identified with a horizontal $\mathfrak{u}(n)$-valued equivariant 1-form on the unitary frame bundle of X restricted to L. In this identification the Maslov class μ defined by L and π may be calculated from $\star_F, \star'_F$ as secondary characteristic class:

Corollary 1.

$$\mu = 2i_L^* \operatorname{tr}_{\mathbb{C}}(\nabla(\star_F) - \nabla(\star'_F)).$$

Indeed, by Proposition 1 the connections are adapted and thus related by local gauge transformations $g : X \supset V \to U(n)$ representing k (see Appendix), hence over L we have (cf. [26])

$$\frac{i}{2\pi} \operatorname{tr}_{\mathbb{C}}(\nabla - \nabla') = \frac{i}{2\pi} \operatorname{tr}(g^{-1} dg) = \frac{i}{2\pi} d \ln \det g = \frac{1}{2} (\det{}^2 g)^* d(\ln : e^z \mapsto z).$$

Note that the Liouville class cannot be extracted likewise as characteristic class in general, but if $S = \tau_{\geq 3} H$ and $\nabla'_F = e^{\frac{i}{\lambda} \operatorname{ad} S} \nabla_F e^{-\frac{i}{\lambda} \operatorname{ad} S}$, one calculates $\omega(\nabla - \nabla') = \delta \tau^3 H = L_{\mathcal{X}H}(\nabla)$.

5. Preliminary analogues to symbol calculus of FIOs

Recall (cf. [7]) that in generalization of (graphs of) symplectomorphisms a canonical relation Λ is defined as Lagrangian submanifold of $X' \times \overline{X}$, where $\overline{(X, \omega)} := (X, -\omega)$ denotes the symplectic conjugated space. The composition

$$\Lambda_1 \circ \Lambda_2 = \Lambda_1 \times_{X'} \Lambda_2 = \pi_{14}(\pi_{12}^{-1} \Lambda_1 \cap \pi_{34}^{-1} \Lambda_2)$$

(here the π_{ij} denote the canonical projections of $X'' \times \overline{X}' \times X' \times \overline{X}$ onto the is and js factor) may then be identified with the image of $\Lambda_1 \times \Lambda_2$ under symplectic reduction of $X'' \times \overline{X}' \times X' \times \overline{X}$ with respect to the canonical coisotropic manifold $C := X'' \times \Delta \times \overline{X}$. Λ_1, Λ_2 are called composable if C intersects $\Lambda_1 \times \Lambda_2$ cleanly, then the product is an immersed Lagrangian submanifold L. Since multiple points of the immersion correspond to multiple intersections of L with some fiber of the characteristic foliation $\pi_{14}|C$ of C, it will be an embedding if the closure of L intersects any fiber at most once.

In terms of the corresponding function algebra $\mathcal{A} := C^\infty(X)$, the above fiber product corresponds either to the topological tensor product

$$\frac{\mathcal{A}'' \hat{\otimes} \mathcal{A}'^{\mathrm{op}}}{\mathcal{I}_{\Lambda_1}} \hat{\otimes}_{\mathcal{A}'} \frac{\mathcal{A}' \hat{\otimes} \mathcal{A}^{\mathrm{op}}}{\mathcal{I}_{\Lambda_2}}$$

or, in terms of symplectic reduction, to

$$N(\mathcal{I}_C)/\mathcal{I}_C + \mathcal{I}_{\Lambda_1 \times \Lambda_2},$$

where $N(\mathcal{I}_C)$ is the Poisson normalizer of the vanishing ideal of C, which consists of functions constant along the fibers of the characteristic foliation $\pi_{14}|C$.

The deformed analogue of a canonical relation Λ will then be a $\star' \otimes \star^{\mathrm{op}}$-module structure on some flat formal line bundle over Λ, which is of the form can $\circ\, S \otimes \phi$ for some Λ-adapted star product $S(\star' \otimes \star^{\mathrm{op}})$ and some flat line bundle ϕ over L, where the equivalence S is nontrivial unless Λ is itself a product. In the case of graphs $\Lambda = \operatorname{graph} \psi$, it was observed by [5, Prop. 3.1] that modulo phase these bimodules yield not more than homomorphisms of star products deforming ψ.

Now, the class of such bimodules is in general unstable under composition, i.e. modulo phases the tensor products will in general collapse to the classical case. Indeed, since we cannot expect a complete symbol calculus of quantized symplectomorphisms by [27], one has to change the category or select composable objects.

1. The most direct way to do so is to consider composable elements at the level of adapted equivalence classes. Given formal deformations $\star_i$ of symplectic manifolds X_i, composable canonical relations $\Lambda_i \subset X_i \times \overline{X}_{i+1}$ with quantizations $*_i = S_i(\star_i \otimes \star_{i+1}^{\mathrm{op}})$ adapted to Λ_i, by $i_\Delta^*[\star \otimes \star^{\mathrm{op}}] = 0$ one can always find de Rham representatives of $[*_1 \otimes *_3]$ whose restrictions to C are basic, i.e. vanish along the fibers of $\pi_{14}|C$. Suppose that this is true as well for the relative class, i.e.

$$i_C^*([*_1 \otimes *_2]_{\Lambda_1 \times \Lambda_3}) \in \pi^* H^2_{dR}(X_1 \times X_4, \Lambda_1 \circ \Lambda_3)(\!(\lambda)\!),$$

where $\pi := \pi_{14}|(C, C \cap \Lambda_1 \times \Lambda_3)$. Then analogously to [5, ch. 5] one can construct an adapted equivalent star product $\star$ which is as well adapted to C, which means that $\mathcal{I}_C := \mathcal{I}_C[\![\lambda]\!]$ is a $\star$-left ideal and its Poisson normalizer $N(\mathcal{I}_C)$ a $\star$-subalgebra. Then the induced star product $(N(\mathcal{I}_C)/\mathcal{I}_C, *')$ is adapted to $\Lambda_1 \circ \Lambda_3$ with class determined by

$$i_C^*([*_1 \otimes *_2]_{\Lambda_1 \times \Lambda_3}) = \pi^*([*']_{\Lambda_1 \circ \Lambda_2}),$$

since $\pi^* \mathcal{I}_{\Lambda_1 \circ \Lambda_3} = i_C^*(\mathcal{I}_{\Lambda_1 \times \Lambda_3} \cap N(\mathcal{I}_C))$.

2. Another strategy is to find some symbol calculus in the derived category of Lagrangian modules. The latter was originally considered in the holomorphic case, first for the sheaf of (micro)differential operators (cf. [17, ch. 5]) and recently for a complex analogue of deformation quantization in [20]. However, in the formal real case, λ-convergence problems remain, this corresponds to the suggestion of Nest and Tsygan in [25] to modify the localization procedure. In case of a cotangent bundle T^*Q one can again consider fiberwise polynomial algebras over $\mathbb{C}(\!(\lambda)\!)$, then the derived category of those modules supported on exact sections which intersect pairwise transversally is stated in [6,19] to be A_∞ equivalent to the Fukaya category "quantizing" the Morse complex on Q. This relates $\star$ representations to mirror symmetry.

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Appendix. Sketch of the symbol calculus of oscillatory distributions

Nice detailed expositions of this theory mainly due to Hörmander [16] are [7,9] for the semiclassical and [12] for the conic (λ-free) case.

A.1. Generating functions

Recall that the image of a section η of $T^*Q \xrightarrow{\pi} Q$ is Lagrangian if and only if $d\eta = 0$, i.e. η is locally the differential of some function on Q, since $\eta^*\theta = \eta$ for the canonical form $\theta = T^*\pi$ on T^*Q. A Lagrangian manifold $L \subset T^*Q$ with caustics (i.e. critical values of $\pi|L$) now can be locally obtained as well from functions ϕ on $B := Q \times \mathbb{R}^k$ (or any surjective submersion $B \xrightarrow{\rho} Q$) via the image L_ϕ of $\operatorname{im} d\phi$ under symplectic reduction of T^*B with respect to the annulator of the vertical bundle $\ker(TB \to \rho^*TQ)^0$, which is given in coordinates as

$$L_\phi = \left\{ \left(q, \frac{\partial \phi}{\partial q}(q, \xi) \right) \middle| \; \frac{\partial \phi}{\partial \xi}(q, \xi) = 0 \right\}. \tag{16}$$

If $\operatorname{im} d\phi$ and C intersect cleanly, L_ϕ is the immersion i_ϕ of the fiber critical set $\Sigma_\phi := \{\frac{\partial \phi}{\partial \xi} = 0\}$ such that caustics correspond to degenerations $\det \frac{\partial^2 \phi}{\partial \xi^2} = 0$, which allows us to attack their local classification by considering some equivalent ϕ as unfolding of $\xi \mapsto \phi(0, \xi)$, cf. [1]. In fact, as proved by Hörmander, the choice of ϕ is locally unique up to (strict) automorphisms of B, additions of constants and direct addition of quadratic forms, cf. [7, Th. 4.18]. Globally, [21] claims that, besides the Liouville class $[i_L^*\theta]$ occurring already for Lagrangian sections, the obstruction to find some $\phi \in C^\infty(Q \times \mathbb{R}^n)$ yielding $L = L_\phi$ is given by the $K^1(L)$ homotopy class defined by the "difference" $k : L \to U(n)/O(n)$ of TL and $T^*_L Q$, as the latter define sections of the bundle of Lagrangian subspaces of $T_L T^*Q$ isomorphic to $L \times U(n)/O(n)$. In particular, the winding number $\mu.[\gamma] := \deg[\det^2 \circ k \circ \gamma]$ associated to some loop $\gamma : S^1 \to L$ is called its Maslov index, and μ the Maslov class of L. Thus, for instance, the circle in $\mathbb{C}$ (harmonic oscillator) does not admit a single generating function (note that this obstruction cannot be circumvented by replacing $\mathbb{R}^n$ by tori, as the first case covers the second in any sense).

Another example to have in mind for the theory of Fourier integral operators is the classical action functional $S(\gamma) = \int_0^1 L(\dot\gamma)\mathrm{d}t$ on the fibration which maps a convenient manifold of free paths $[0, 1] \to Q$ to their ends in $Q \times Q$. If the corresponding Hamiltonian system given by the Legendre transform of L is complete, then L_S is (up to symplectic conjugation given by momenta inversion $\overline{(q, p)} = (q, -p)$ in T^*Q) the graph of its time 1 flow.

A.2. Oscillatory distributions

Basic elements of short wave asymptotics are the the WKB-waves, i.e half densities of type $e^{-iS/\lambda}a_\lambda$ with phase $S \in C^\infty(Q)$ and a_λ a λ power series of half densities on Q. Their key property is given by the stationary phase formula: If $\mathbb{R}^k \ni \xi \mapsto \phi(q, \xi)$ is a family of WKB phases such that L_ϕ has no caustics, then their superposition $I(\phi, a)(q) := \frac{e^{i\pi k/4}}{(2\pi\lambda)^{k/2}} \int_{\xi\in\mathbb{R}^k} e^{i\phi(q,\xi)/\lambda} a(q, \xi) d^k\xi$ is λ-asymptotically equivalent to the WKB wave

$$I(\phi, a) \sim e^{\frac{i}{\lambda}\phi} e^{-\frac{i\pi}{2}\,\mathrm{ind}\,\partial^2_\xi\phi} \left(\frac{a}{\sqrt{|\det \partial^2_\xi \phi|}} + O(\lambda) \right) \circ \rho|^{-1}_{\Sigma_\phi}. \tag{17}$$

where $\phi \circ \rho|^{-1}_{\Sigma_\phi}$ coincides with the enveloping phase (Huygens' principle). It follows that if the composition $\int_Q I(\phi, a) I(\psi, b)$ is well defined, then its asymptotics may be written as sum (integral) over the intersection points $L_\phi \cap L_\psi = \{d(\phi - \psi) = 0\}$ in the case of transversal (clean) intersections, which allows us to lift the singular support of such distributions to their microsupport $WF(I(\psi, a)) := \mathrm{supp}(a \circ i_\psi^{-1}) \subset \overline{L_\psi}$ in phase space T^*Q.

Indeed, via $\pi|L^{-1}$ the development (17) may be naturally identified with a half density on L such that the singularities of the denominator in (17) at caustics appear as artefact of the projection $\pi|L$ onto Q. Moreover, if the microsupports of a set of oscillatory distributions $I(\phi_i, a_i)$ all lie inside some single Lagrangian L, then the differences of the pulled back phases in (17) define locally constant Čech 1 cocycles on L corresponding to the class (1). In summary, the leading asymptotics of oscillatory distributions $O(L)$ microsupported on L are described by constant sections of the flat complex line bundle

$$|\wedge|^{\frac{1}{2}} L \otimes \exp\left(\frac{i}{\lambda} i_L^*[\theta] - \frac{i\pi}{2}\mu \right)$$

called principal symbols. In particular, the canonical isomorphisms $\overline{O(L)} = O(\overline{L})$ and $O(L \times L') = O(L)\hat\otimes O(L')$ allow us to speak of distributions microsupported on canonical relations $L \subset T^*Q \times \overline{T^*Q}$, which per definition provide the kernels of Fourier integral operators (FIOs). Then the composition of FIOs, if well defined, corresponds to the naturally defined composition of principal symbols, cf. [7, ch.6].

A.3. Reminder on deformation quantization

In particular, the kernels of pseudodifferential operators are oscillatory distributions microsupported on the identity (i.e., the conormal bundle of the diagonal in $T^*(Q \times Q)$), which is naturally identified with T^*Q. Then star products arise as asymptotics of their composition. These products were defined purely algebraically for any symplectic (or Poisson) manifold (X, ω) in [2] (see also the survey articles [11,15]) as a formal deformation $(C^\infty(X)[\![\lambda]\!], \star)$ of the classical algebra $(C^\infty(X), \cdot)$ by a sum of bidifferential operators

$$\star := \sum_{i=0}^{\infty} \lambda^i \star_i \tag{18}$$

with $\star_0 = \cdot$, such that

i. $\star$ is associative $[\star, \star] = 0$, which may be written order by order as

$$2b \star_n + \sum_{i=1}^{n-1} [\star_i, \star_{n-i}] = 0, \tag{19}$$

where b, $[.,.]$ are the Hochschild coboundary[2] and Gerstenhaber bracket[5] of $\mathcal{C}(C^\infty(X), C^\infty(X))$, respectively;

ii. the commutator $\frac{i}{\lambda}[.,.]_\star$ deforms the classical Lie structure $\frac{i}{\lambda}[f, g]_* = \{f, g\} \bmod O(\lambda)$, thus by (22) we have

$$\star_1 = \frac{i}{2}\{.,.\} + bS_1 \tag{20}$$

iii. $1 \star f = f \star 1 = f$.

Two such deformations $\star, \star'$ are considered equivalent if they are linked by some algebra isomorphism S given by a series of differential operators $S = id + \sum_{i=1}^{\infty} \lambda^i S_i$; this is denoted as $\star' = S(\star) := S^{-1} \circ \star \circ S \otimes S$. In the case of a symplectic manifold (X, ω) the equivalence classes are in bijection with $\lambda^{-1}[\omega] + H^2_{dR}(X)[[\lambda]]$, see [24, App.] for a short demonstration or [11] for more references.

The equivalence of two star products $\star, \star'$ in case $H^2_{dR}(X) = 0$ was first observed by Lichnerowicz: By applying $1 + \lambda S_1$ with S_1 as in condition (20), we may assume that $\star = \star' = \frac{i}{2}\{.,.\}$. If now $\star = \star' \bmod O(\lambda^k)$, then by (19)

$$b(\star_k - \star'_k) = 0 \quad \text{and} \quad b(\star_{k+1} - \star'_{k+1}) + [\star_1, \star_k - \star'_k] = 0. \tag{21}$$

Now, antisymmetrization and ω provide isomorphisms

$$H\mathcal{C}(C^\infty(X); C^\infty(X)) \cong \Gamma\left(\bigwedge TX\right) \cong \Omega(X) \tag{22}$$

calculating the Hochschild cohomology[2] of $C^\infty(X)$ (Kostant–Hochschild–Rosenberg (HKR) theorem), on which $\mathrm{ad}(\star_1)$ acts as de Rham coboundary d; hence by $H^2_{dR}(X) = 0$ it follows $\star_k - \star'_k = -\mathrm{ad}(\star_1)S_k$ for some derivation $S_k \in \ker b$. Thus $id + \lambda^k S_k$ provides the induction step for constructing an equivalence between $\star$ and $\star'$.

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[5] The Gerstenhaber bracket on $\mathcal{C}$ is the graded supercommutator of the product $C \circ C'(f_0, \ldots, f_{k+l}) := \sum_{i=0}^{k} (-1)^{il} C(f_0, \ldots, f_{i-1}, C'(f_i, \ldots, f_{i+l}), f_{i+l+1}, \ldots, f_{k+l})$ turning $\mathcal{C}$ into a differential graded super Lie algebra w.r.t. the degree $|\mathcal{C}^k| := k$.

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